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Potential scattering in a slowly varying laser field: relativistic generalizations

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Abstract. A variational approach to the problem of relativistic potential scattering in a laser field is developed. The field is assumed to be slowly varying compared with the collision time and to have a well defined direction of propagation, but is otherwise arbitrary. A trial function is chosen having the correct gauge-transformation property and accounting in an approximate manner for the simultaneous interaction of the projectile with the centre of force and with the laser field. The variational calculation provides a low-frequency approximation for the transition amplitude of improved accuracy compared with those obtained in previous treatments of this problem. The analysis is first given in terms of a spin-zero wave equation and then extended to allow for the scattering of a Dirac particle. A relativistic analogue of the Kroll-Watson approximation is obtained when the field is taken to be monochromatic and certain higher-order correction terms are dropped.

1. Introduction

A low-frequency approximation for the relativistic scattering of two scalar particles, one charged and the other neutral, in the presence of a strong, monochromatic external field, was derived by Brown and Goble (1968) using very general invariance and analyticity considerations. The approximation involves as input the field-free scattering cross section and, remarkably, correctly includes the first *two* terms in an expansion in powers of the frequency. In a subsequent elaboration of this approach (Rosenberg 1982) the result appropriate to a monochromatic field was generalized to allow for an arbitrarily polarized wave train of finite extent, and for a charged particle of spin-$\frac{1}{2}$. This latter paper also contained a demonstration of the close analogy which exists between the Brown–Goble approximation and that derived by Kroll and Watson (1973) in the context of non-relativistic potential scattering. More recently, a low-frequency approximation was derived by Kaminski (1985), applicable to the potential scattering of a Dirac particle in a monochromatic, linearly polarized field. That result gave what might be called the leading term of an approximation of the Kroll–Watson type; it did not include the first-order correction terms which one expects (from the above-mentioned analogy, for example) should be present. It may be anticipated that these corrections will take the form of small shifts in the momenta appearing as arguments of the on-shell field-free scattering amplitude. Here we adopt a variational approach which does in fact generate first-order correction terms of this type; these terms reduce,

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in the non-relativistic limit, to the expected form, and provide recoil corrections of order v/c not present in the Kroll–Watson formula. Moreover, the calculation provides an explicit expression, involving the field-free wavefunction, for the second- and third-order corrections as well. (We will be more precise later on in specifying levels of accuracy.) As with any variational calculation, the essential feature is the choice of trial function; it is constructed here by generalizing to the relativistic case the approximate wavefunction used (non-variationally) by Kroll and Watson. The suggestion that the Kroll–Watson wavefunction would be useful as a trial function in a variational calculation (for non-relativistic scattering in a laser field) was made by Kaminski (1988). Such an approach appears to provide an effective procedure for developing low-frequency approximations applicable to a variety of scattering (and ionization) problems. We confine our attention to relativistic potential scattering here and will report on other applications in the future.

2. Variational principle

2.1. Modified plane waves

We first consider the scattering of a particle of zero spin. The wave equation in the absence of the external field is taken to be not the usual Klein–Gordon equation, but rather a linearized version (in units with $\hbar = c = 1$)

$$(\sqrt{-\nabla^2 + m^2} + V(x))u(x, t) = \mathrm{i}\frac{\partial}{\partial t}u(x, t). \tag{2.1}$$

As shown by Goldberger and Watson (1964), the solutions of (2.1) will differ only little from those of the Klein–Gordon equation if the potential is weak enough so that pair creation effects are negligible and is slowly varying relative to the de Broglie wavelength. Equation (2.1) is preferred here since, being of first order in the time derivative, it is closer in form to the Dirac equation; the extension of the theory to the spin-$\frac{1}{2}$ case (taken up in section 3 below) will then be quite straightforward. The external field is assumed to have a well defined direction of propagation, taken along the z axis. It is described by a 4-vector potential $A = (A_x, A_y, A_z, A_0)$, a function of the variable $\tau = t - z$ (not to be confused with 'proper time'), and satisfying the Lorentz condition $\partial A = 0$. We then have $nA = 0$, where n has components $n_z = n_0 = 1$ with the others vanishing. (The metric is such that $ab = \boldsymbol{a}\cdot\boldsymbol{b} - a_0 b_0$.) It is sometimes convenient to work in the Coulomb gauge, with $A_z = A_0 = 0$ and $\boldsymbol{A}$ equal to a divergenceless 3-vector $\boldsymbol{A}_C$ lying in the xy plane. More generally, there is a (gauge) freedom in the choice of the function $A_z = A_0$ and the requirement that all calculated cross sections be invariant with respect to changes in this arbitrary function provides a useful guide in the construction of approximations.

The wave equation in the absence of the scattering potential, but with the field present, is obtained from the minimal-coupling prescription as

$$\sqrt{(-\mathrm{i}\nabla - e\boldsymbol{A})^2 + m^2}\,\varphi = \left(\mathrm{i}\frac{\partial}{\partial t} - eA_0\right)\varphi. \tag{2.2}$$

This equation has an exact solution (Volkov 1935) of the form

$$\varphi_p(x) = (2\pi)^{-3/2}\exp\left(-\mathrm{i}\int_0^\tau I_p(\tau')\,\mathrm{d}\tau' + \mathrm{i}px\right) \tag{2.3}$$

with

$$I_p(\tau)=\frac{1}{2np}[2epA(\tau)-e^2A^2(\tau)]. \tag{2.4}$$

(The presence of a constant phase factor multiplying the pure plane wave in the limit $t\to-\infty$ in the solution (2.3) is of no physical consequence.) To verify that the wave equation is satisfied, one inserts the expression (2.3) into (2.2) and (letting $\hat{z}$ represent a unit vector in the z direction) arrives at the condition

$$\sqrt{(\boldsymbol{p}-e\boldsymbol{A}+\hat{z}I_p)^2+m^2}=p_0-eA_0+I_p\equiv p_0(\tau) \tag{2.5}$$

which holds for $p^2+m^2=0$ and $nA=0$. Note that under the gauge transformation $A\to A-n\,\mathrm{d}\lambda(\tau)/\mathrm{d}\tau$ the corresponding transformation of the Volkov solution is

$$\varphi_p\to\exp\{ie[\lambda(\tau)-\lambda(0)]\}\varphi_p.$$

This conforms with the gauge invariance of the theory; the presence of the constant phase involving $\lambda(0)$ is of no consequence in this regard.

2.2. Trial scattering wavefunction

The wave equation, in the presence of both the scattering potential and the external field, is written as

$$\sqrt{(-\mathrm{i}\boldsymbol{\nabla}-e\boldsymbol{A})^2+m^2}\,\psi=\left(\mathrm{i}\frac{\partial}{\partial t}-eA_0-V\right)\psi. \tag{2.6}$$

The exact outgoing-wave solution corresponding to an incident wave of momentum p reduces to the solution of (2.1) in the absence of the field; i.e.

$$\psi_p^{(+)}(x)\to\exp(-\mathrm{i}p_0t)u_p^{(+)}(\boldsymbol{x})\qquad\text{for } A\to0$$

where the field-free scattering function satisfies

$$\sqrt{-\nabla^2+m^2}\,u_p^{(+)}(\boldsymbol{x})=(p_0-V)u_p^{(+)}(\boldsymbol{x}). \tag{2.7}$$

In the absence of the scattering potential the exact solution reduces to the Volkov wavefunction. We now introduce a trial function, satisfying these boundary conditions, of the form

$$\tilde{\psi}_p^{(+)}(x)=\exp[-\mathrm{i}\Phi_p(\tau)]u_{p(\tau)}^{(+)} \tag{2.8a}$$

with

$$\Phi_p(\tau)=\int_0^\tau I_p(\tau')\,\mathrm{d}\tau'-e\boldsymbol{A}(\tau)\cdot\boldsymbol{x}+I_p(\tau)z+p_0t \tag{2.8b}$$

and

$$\boldsymbol{p}(\tau)=\boldsymbol{p}-e\boldsymbol{A}+\hat{z}I_p. \tag{2.8c}$$

The momentum $\boldsymbol{p}(\tau)$ is recognized (Brown and Goble 1968) as the solution of the classical equation of motion for the particle in the field. It is readily verified that the trial function (2.8a) has the correct gauge-transformation property. In the non-relativistic limit, with recoil corrections of order v/c ignored, it reduces to the approximate wavefunction introduced by Kroll and Watson.

A measure of the error in the trial function is obtained by evaluating

$$B \equiv \left(H - \mathrm{i}\frac{\partial}{\partial t} \right) \tilde{\psi}_p^{(+)}(x) \tag{2.9}$$

with

$$H = \sqrt{(-\mathrm{i}\nabla - e\boldsymbol{A})^2 + m^2} + V + eA_0. \tag{2.10}$$

With the aid of (2.5) and the version of (2.7) in which $\boldsymbol{p}$ and p_0 are replaced by $\boldsymbol{p}(\tau)$ and $p_0(\tau)$, respectively, we find that

$$B = \mathrm{e}^{-\mathrm{i}\Phi} \left(\sqrt{(-\mathrm{i}\nabla + e\hat{z}\boldsymbol{F}\cdot\boldsymbol{x})^2 + m^2} - \sqrt{-\nabla^2 + m^2} - e\boldsymbol{F}\cdot\boldsymbol{x} - \mathrm{i}\frac{\partial}{\partial t} \right) u_{p(\tau)}^{(+)} \tag{2.11}$$

where, in terms of the electric field $\boldsymbol{E}$, we have defined an effective field vector

$$\boldsymbol{F}(\tau) \equiv \boldsymbol{E}(\tau) - \hat{z}\boldsymbol{E}(\tau)\cdot\frac{\boldsymbol{p}(\tau)}{np}. \tag{2.12}$$

We may interpret $\boldsymbol{W} \equiv -\hat{z}\boldsymbol{F}\cdot\boldsymbol{x}$ as a relativistic generalization of the vector potential in the length gauge. The first two terms on the right-hand side of (2.11) have the structure (with $\boldsymbol{p} = -\mathrm{i}\nabla$ and $\varepsilon = (-\nabla^2 + m^2)^{1/2}$)

$$\sqrt{(\boldsymbol{p} - e\boldsymbol{W})^2 + m^2} - \varepsilon \approx \varepsilon^{-1/2} \left(-\frac{e}{2}(\boldsymbol{p}\cdot\boldsymbol{W} + \boldsymbol{W}\cdot\boldsymbol{p}) \right) \varepsilon^{-1/2} \tag{2.13}$$

correct to first order in the electric field. In view of the form (Goldberger and Watson 1964)

$$\boldsymbol{j}(\boldsymbol{x}) = \frac{1}{2\mathrm{i}} [(\varepsilon^{-1/2}u^*)(\varepsilon^{-1/2}\nabla u) - (\varepsilon^{-1/2}\nabla u^*)(\varepsilon^{-1/2}u)] \tag{2.14}$$

taken by the conserved current associated with the wave equation (2.1), one may view the expression on the right-hand side of (2.13) as representing the interaction of the conserved current with the vector potential $\boldsymbol{W}$. Turning our attention to the last term in (2.11), we note that

$$\frac{\partial}{\partial t} u_{p(\tau)} = (\nabla_{p(\tau)} u_{p(\tau)}) \cdot \frac{\partial \boldsymbol{p}(\tau)}{\partial t} \tag{2.15a}$$

and

$$\frac{\partial \boldsymbol{p}(\tau)}{\partial t} = e\boldsymbol{F}(\tau). \tag{2.15b}$$

(If one were to calculate the *total* derivative of the classical momentum, then $\boldsymbol{F}$ on the right-hand side would of course be replaced by $\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}$; the two field vectors differ by a factor $\mathrm{d}\tau/\mathrm{d}t = 1 - v_z(\tau)$.) Thus the error in the trial function, as indicated by (2.11)-(2.15), is of the order of the electric field (rather than the vector potential), in agreement with the analogous property (Kaminski 1988) of the Kroll-Watson wavefunction. (The origin of this property may be traced to the form of the phase function (2.8*b*), which essentially brings about a transformation from the momentum to the length gauge.) It follows that a variational approximation will contain an error of *second* order in the field strength. If the motion of the particle in the neighbourhood of the centre of force is perturbed only slightly by the presence of the external field

(as will be true in a wide variety of cases of interest in atomic and nuclear applications) we may expect the variational approach to be an effective one. Since the electric field is a time-derivative of the vector potential, the approximation procedure should be particularly appropriate for slowly varying fields. (A low-frequency field, even of moderate intensity, will have a substantial effect on the motion of the particle in initial and final states; this is accounted for non-perturbatively in the trial function (2.8).)

2.3. Variational approximation

Following standard methods, we may take as our starting point the expression

$$S(p', p) = \mathrm{i} \lim_{t' \to \infty} \lim_{t \to -\infty} \int \mathrm{d}^3x' \int \mathrm{d}^3x \, \varphi_{p'}^*(x') G(x', x) \varphi_p(x) \tag{2.16a}$$

for the S matrix in terms of the Green function G which effects the time evolution of the system. Alternative versions of this representation are

$$S(p', p) = \lim_{t \to \infty} \int \mathrm{d}^3x \, \varphi_{p'}^*(x) \psi_p^{(+)}(x) \tag{2.16b}$$

and

$$S(p', p) = \lim_{t \to -\infty} \int \mathrm{d}^3x \, \psi_{p'}^{(-)*}(x) \varphi_p(x) \tag{2.16c}$$

where the superscript $(-)$ signifies incoming-wave boundary conditions. Use of the Lippmann–Schwinger integral equation for the exact solution in (2.16*b*) leads to the identity

$$S(p', p) = \delta(\boldsymbol{p}' - \boldsymbol{p}) - \mathrm{i} T(p', p) \tag{2.17}$$

with

$$T(p', p) = \int \mathrm{d}^4x \, \varphi_{p'}^* V \psi_p^{(+)}. \tag{2.18}$$

A variational identity for the transition amplitude may be written in the form

$$T(p', p) = \int \mathrm{d}^4x \left[\varphi_{p'}^* V \tilde{\psi}_p^{(+)} + \psi_{p';\mathrm{sc}}^{(-)*} \left(H - \mathrm{i} \frac{\partial}{\partial t} \right) \tilde{\psi}_p^{(+)} \right] \tag{2.19}$$

with the incoming scattered wave defined as

$$\psi_{p';\mathrm{sc}}^{(-)} \equiv \psi_{p'}^{(-)} - \varphi_{p'}. \tag{2.20}$$

This scattered wave vanishes in the distant future; the outgoing scattered wave satisfies the analogous boundary condition

$$\psi_{p;\mathrm{sc}}^{(+)} \equiv \psi_p^{(+)} - \varphi_p \to 0 \qquad t \to -\infty. \tag{2.21}$$

Strictly speaking, these conditions hold when the asymptotic plane waves, adopted here to simplify the presentation, are replaced by localized wavepackets. When that is done the trial function shown in (2.8), which (as noted above) has the property of reducing to the Volkov wavefunction in the absence of the scattering potential, will satisfy the same boundary condition (2.21) as does the exact solution. The demonstration that the right-hand side of (2.19) does in fact represent the T matrix as defined

in (2.18) is achieved through an integration by parts (with H taken to be Hermitian) and the use of the boundary conditions discussed above, along with the wave equation satisfied by the exact incoming scattered wave. If this function is replaced by an approximation to it, differing by a first-order quantity, one sees that the error thereby introduced in the transition amplitude is of *second* order, assuming that the quantity B of (2.9) is of first order. We now begin an examination of the nature of such a variational approximation.

With the trial function chosen as in (2.8) the first term in (2.19) may be written, after a change of independent variables to the set (x, y, z, τ), as

$$T^{(1)} = \int_{-\infty}^{\infty} d\tau \exp\left(i \int_0^{\tau} d\tau' [I_{p'}(\tau') - I_p(\tau')] + i(p_0' - p_0)\tau\right) t[\boldsymbol{Q}(\tau), \boldsymbol{p}(\tau)] \tag{2.22a}$$

where, with $\boldsymbol{p}'(\tau) = \boldsymbol{p}' - e\boldsymbol{A}(\tau) + \hat{z} I_{p'}(\tau)$, we have defined

$$\boldsymbol{Q}(\tau) = \boldsymbol{p}'(\tau) - \hat{z}[I_{p'}(\tau) - I_p(\tau) + p_0' - p_0]. \tag{2.22b}$$

The off-shell field-free t matrix is represented as

$$t(\boldsymbol{p}', \boldsymbol{p}) = (2\pi)^{-3/2} \int d^3x \exp(-i\boldsymbol{p}' \cdot \boldsymbol{x}) V u_{\boldsymbol{p}}^{(+)}(\boldsymbol{x}). \tag{2.22c}$$

The use of a trial incoming scattered wave in place of the exact function in the second term in (2.19) provides a first-order correction to $T^{(1)}$. With the choice

$$\tilde{\psi}_{p';\mathrm{sc}}^{(-)} = \exp[-i\Phi_{p'}(\tau)] u_{\boldsymbol{p}'(\tau);\mathrm{sc}}^{(-)}$$

and with the aid of (2.11), (2.12) and (2.15) this term may be put in the form

$$T^{(2)} = \int_{-\infty}^{\infty} d\tau \exp\left(i \int_0^{\tau} d\tau' [I_{p'}(\tau') - I_p(\tau')] + i(p_0' - p_0)\tau\right) w(\tau) \tag{2.23a}$$

with

$$w(\tau) = \int d^3x \exp i[p_0' - p_0 + I_{p'}(\tau) - I_p(\tau)] z \times u_{\boldsymbol{p}'(\tau);\mathrm{sc}}^{(-)*}[(-i\nabla_{\bar{p}(\tau)} - \underline{x}) \cdot e\underline{F} + \sqrt{(-i\nabla + e\hat{z}\underline{F} \cdot \underline{x})^2 + \underline{m}^2} - \sqrt{-\nabla^2 + \underline{m}^2}] u_{\bar{p}(\tau)}^{(+)}. \tag{2.23b}$$

As indicated earlier in connection with (2.13) and (2.14), the square-root terms in (2.23b) can be interpreted (to first order in the electric field) as the interaction of the off-diagonal matrix element of the conserved current with the vector potential $-\hat{z}\boldsymbol{F} \cdot \boldsymbol{x}$. In section 4, below, we examine a number of special cases to gain further insight into the nature of this variational approximation. First, however, we indicate how the formalism may be generalized to include spin degrees of freedom.

3. Dirac equation

The analysis will now be extended to apply to the scattering of a Dirac particle. In the absence of the field the wave equation is

$$[\boldsymbol{\alpha} \cdot (-i\nabla) + \beta m + V(\boldsymbol{x})] u(\boldsymbol{x}, t) = i \frac{\partial}{\partial t} u(\boldsymbol{x}, t). \tag{3.1}$$

(For simplicity we ignore the effect of an anomalous magnetic moment, and we suppress all spin variables in the specification of the wavefunctions.) The equation which generates the Volkov modified plane wave is

$$[\boldsymbol{\alpha}\cdot(-\mathrm{i}\boldsymbol{\nabla}-e\boldsymbol{A})+\beta m]\varphi=\left(\mathrm{i}\frac{\partial}{\partial t}-eA_0\right)\varphi \tag{3.2}$$

a solution for which we seek in the form

$$\varphi_p(x)=(2\pi)^{-3/2}\exp\left(-\mathrm{i}\int_0^\tau \mathrm{d}\tau'\, I_p(\tau')+\mathrm{i}px\right)\chi_p. \tag{3.3}$$

When this form is substituted into (3.2) we find, as the equation determining χ,

$$\left(p_0(\tau)+\mathrm{i}\frac{\partial}{\partial t}\right)\chi_p=[\boldsymbol{\alpha}\cdot\boldsymbol{p}(\tau)+\beta m+\boldsymbol{\alpha}\cdot(-\mathrm{i}\boldsymbol{\nabla})]\chi_p \tag{3.4}$$

with

$$p(\tau)=p-eA(\tau)+nI_p(\tau) \tag{3.5}$$

as previously defined. It may be verified that a solution to (3.4) is provided by

$$\chi_p(\tau)=\left(1+\frac{e}{2np}(\alpha_z+1)(\boldsymbol{\alpha}\cdot\boldsymbol{A}-A_0)\right)\Psi(p) \tag{3.6}$$

where $\Psi(p)$, for $p^2+m^2=0$, satisfies

$$(\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m)\Psi(p)=p_0\Psi(p). \tag{3.7}$$

As a first step in verifying (3.6) one uses the relations

$$\alpha_z(\alpha_z+1)=\alpha_z+1 \qquad -\frac{\partial}{\partial z}\chi_p(\tau)=\frac{\partial}{\partial t}\chi_p(\tau)$$

to show that the terms involving the space and time derivatives cancel in (3.4). The equation for χ then becomes

$$[\boldsymbol{\alpha}\cdot\boldsymbol{p}(\tau)+\beta m]\chi_p=p_0(\tau)\chi_p. \tag{3.8}$$

To demonstrate that this equation is satisfied by the expression (3.6), it is convenient to work with its alternative covariant form

$$\chi_p=\left(1-\frac{e}{2np}(\gamma n)(\gamma A)\right)\Psi(p) \tag{3.9}$$

with $\gamma_i=\beta\alpha_i$, $\gamma_0=\beta$, since the Dirac algebra is simpler when expressed in terms of the matrices γ_μ. The demonstration, which is not reproduced here, makes use of the anticommutation relations satisfied by these matrices along with the relations $nA=(\gamma n)^2=0$. Similar techniques are used to confirm that with $\Psi(p)$ normalized to unity the same is true for χ_p. This allows us to make the identification

$$\chi_p(\tau)=\Psi[p(\tau)]. \tag{3.10}$$

A trial function of the generalized Kroll–Watson form is provided by the expression shown in (2.8), above, with the field-free solution now understood to be that appropriate to the Dirac equation:

$$[\boldsymbol{\alpha}\cdot(-\mathrm{i}\boldsymbol{\nabla})+\beta m+V]u^{(+)}_{p(\tau)}=p_0(\tau)u^{(+)}_{p(\tau)}. \tag{3.11}$$

As in the spin-zero case the trial function correctly reduces to the Volkov solution in the absence of the scattering potential.

The variational approximation is obtained from the identity (2.19) where now

$$H = \boldsymbol{\alpha} \cdot (-\mathrm{i}\boldsymbol{\nabla} - e\boldsymbol{A}) + \beta m + V + eA_0 \tag{3.12}$$

(and the asterisk denotes Hermitian adjoint). Making use of (3.11) we find that

$$\left(H - \mathrm{i}\frac{\partial}{\partial t}\right)\tilde{\psi}_p^{(+)} = \exp[-\mathrm{i}\Phi_p(\tau)][(-\mathrm{i}\boldsymbol{\nabla}_{p(\tau)} - \boldsymbol{x}) \cdot e\boldsymbol{F} + \boldsymbol{\alpha} \cdot e\hat{z}\boldsymbol{F} \cdot \boldsymbol{x}]u_{p(\tau)}^{(+)}. \tag{3.13}$$

The second term on the right represents (in analogy with the corresponding term arising in the spin-zero case) the interaction of the conserved Dirac current with the vector potential in the length gauge. The variational approximation is $T \simeq T^{(1)} + T^{(2)}$, with the first and second terms given by the appropriate spin-$\frac{1}{2}$ versions of (2.22) and (2.23), respectively. Thus the off-shell t matrix in (2.22a) now takes the form

$$t[\boldsymbol{Q}(\tau), \boldsymbol{p}(\tau)] = (2\pi)^{-3/2} \int \mathrm{d}^3x \exp(-\mathrm{i}\boldsymbol{Q}(\tau) \cdot \boldsymbol{x})\chi_p^*(\tau)V(\boldsymbol{x})u_{p(\tau)}^{(+)}(\boldsymbol{x}). \tag{3.14}$$

For later reference we note that this t matrix will be on the mass shell if $\boldsymbol{Q}(\tau)$, defined in (2.22b), is replaced by $\boldsymbol{p}'(\tau) = \boldsymbol{p}' - e\boldsymbol{A} + \hat{z}I_{p'}$. The correction term is of the form (2.23a) where now, in place of (2.23b), we have

$$w(\tau) = \int \mathrm{d}^3x \exp[\mathrm{i}(p_0' - p_0 + I_{p'} - I_p)z]$$
$$\times u_{p'(\tau);\mathrm{sc}}^{(-)*}[(-\mathrm{i}\boldsymbol{\nabla}_{p(\tau)} - \boldsymbol{x}) \cdot e\boldsymbol{F} + \boldsymbol{\alpha} \cdot e\hat{z}\boldsymbol{F} \cdot \boldsymbol{x}]u_{p(\tau)}^{(+)}. \tag{3.15}$$

4. Some limiting cases

4.1. Intermediate- and strong-coupling regimes

The main achievement of the variational method developed here is the determination of the correction term $T^{(2)}$ in the form shown in (2.23). There are circumstances, however, in which one is not interested in maintaining this level of accuracy; it would then be useful to have available simpler, if less accurate, versions of the leading approximation given by (2.22). In looking for such simplifications let us suppose, to begin with, that the field is strong enough or sufficiently slowly varying so that the phase in (2.22a) varies rapidly as a function of τ, much more so than does the t-matrix factor in that equation. We may then apply a stationary-phase approximation to the evaluation of the integral (Rosenberg 1982). At the point of stationary phase we have the equality

$$p_0' - p_0 = I_p - I_{p'}. \tag{4.1}$$

With this condition imposed the final-state momentum in (2.22b) is changed to

$$\boldsymbol{Q} = \boldsymbol{p}' = e\boldsymbol{A} + \hat{z}I_{p'} \tag{4.2}$$

in which case the t matrix in (2.22a) becomes $t[\boldsymbol{p}'(\tau), \boldsymbol{p}(\tau)]$. The initial and final momenta appearing in this expression satisfy

$$|\boldsymbol{p}(\tau)|^2 + m^2 = p_0^2(\tau) \quad \cdot \quad |\boldsymbol{p}'(\tau)|^2 + m^2 = p_0'^2(\tau)$$

respectively. Furthermore, by virtue of (4.1) the energies $p_0(\tau)$ and $p_0'(\tau)$ are equal so that the t matrix in the spin-zero case may be identified with the physical field-free scattering amplitude. To complete this argument for spin-$\frac{1}{2}$ scattering we note that there we require (in addition to the validity of the condition (4.2)) that the final-state spinor satisfies the Dirac equation appropriate to the on-shell momentum $\boldsymbol{p}'(\tau)$:

$$(\boldsymbol{\alpha}\cdot\boldsymbol{p}'(\tau)+\beta m)\chi_{p'}=p_0'(\tau)\chi_{p'} \tag{4.3}$$

which is in fact the case. (The corresponding equation for the initial-state spinor has appeared earlier in (3.8).)

Let us now consider an 'intermediate-coupling' regime (Rosenberg 1982) in which both the amplitude of the vector potential and its characteristic frequency are treated as first-order quantities, and second-order terms in (2.22a) are neglected. In the argument of the t matrix appearing there we write

$$\boldsymbol{Q}(\tau)=\boldsymbol{Q}_0(\tau)-\hat{z}(p_0'-p_0) \tag{4.4}$$

and, with the energy shift taken to be of first order, we expand this function as

$$t[\boldsymbol{Q}(\tau),\boldsymbol{p}(\tau)]\simeq t[\boldsymbol{Q}_0(\tau),\boldsymbol{p}(\tau)]-(p_0'-p_0)\hat{z}\cdot\boldsymbol{\nabla}_{\boldsymbol{Q}(\tau)}t[\boldsymbol{Q}(\tau),\boldsymbol{p}(\tau)]. \tag{4.5}$$

An integration by parts results in the replacement

$$-(p_0'-p_0)\rightarrow I_{p'}-I_p$$

valid to the required accuracy. Here we ignore the contribution arising from the differentiation of the t matrix since that introduces a second-order term in the form of the derivative of the slowly-varying vector potential. The surface terms are assumed to vanish. This may be justified through the introduction of a convergence factor $\exp(-\eta|\tau|)$, with η taken to be a positive infinitesimal which is allowed to vanish at the end of the calculation. The result of these manipulations is the replacement of the final momentum $\boldsymbol{Q}(\tau)$ by $\boldsymbol{p}'(\tau)$ as in (4.2). We cannot yet conclude that the t matrix is on shell since (4.1) has not been verified. The deviation from the mass shell may be measured by expressing the t matrix in terms of the four scalar variables

$$\xi=\sqrt{|\boldsymbol{p}(\tau)|^2+m^2}-p_0(\tau)\qquad\qquad \xi'=\sqrt{|\boldsymbol{p}'(\tau)|^2+m^2}-p_0(\tau)$$

along with the energy and momentum-transfer variables $p_0(\tau)$, and $[\boldsymbol{p}'(\tau)-\boldsymbol{p}(\tau)]^2$, respectively. The t matrix is on shell when $\xi=0$ and $\xi'=0$; while the first condition is satisfied the second is not. However, we note that

$$\xi'=p_0'(\tau)-p_0(\tau)=p_0'+I_{p'}-p_0-I_p.$$

Treating this as a quantity of first order, we may expand the t matrix about $\xi'=0$ and discard second-order corrections. The first-order contribution vanishes, as may be seen by once again applying the integration-by-parts procedure. It follows that only on-shell values of the t matrix need be known in the evaluation of the first-order approximation (2.22a) in the intermediate-coupling regime.

It may be appropriate to emphasize that the value of the components $A_0=A_z$ of the vector potential may be chosen arbitrarily, so that a characterization of the strength of the potential (as is done above in defining intermediate- and strong-coupling regimes) is meaningful only in a gauge-invariant formulation such as the one developed here. In what follows we make the simplest choice $A_0=A_z=0$.

4.2. Monochromatic field

We now specialize to a linearly-polarized monochromatic field with vector potential chosen, in the Coulomb gauge, to be $\boldsymbol{A}_C = \boldsymbol{a}\cos\omega\tau$. In (2.22*a*) we write

$$I_p(\omega\tau) = \frac{1}{2np}(2e\boldsymbol{p}\cdot\boldsymbol{a}\cos\omega\tau - \tfrac{1}{2}e^2a^2\cos 2\omega\tau) - \frac{e^2a^2}{4np} \tag{4.6}$$

along with a similar expression involving the final momentum. (In order to simplify subsequent formulae we have taken the liberty of changing notation, with $\omega\tau$ rather than τ now chosen as the independent variable.) After the introduction of a Fourier expansion, the integration over τ may be completed to yield

$$T^{(1)} = 2\pi\sum_{l=-\infty}^{\infty}\delta\left(p_0' - p_0 - \frac{e^2a^2}{4np'} + \frac{e^2a^2}{4np} - l\omega\right)T_l^{(1)}. \tag{4.7}$$

The coefficients are obtained in the form

$$T_l^{(1)} = \int_0^{2\pi}\frac{d\theta}{2\pi}\exp[il\theta + i(\rho'-\rho)\sin\theta - i(\alpha'-\alpha)\sin 2\theta]t[\boldsymbol{Q}(\theta), \boldsymbol{p}(\theta)] \tag{4.8}$$

where we have defined

$$\rho = \frac{e\boldsymbol{p}\cdot\boldsymbol{a}}{\omega np} \qquad \alpha = \frac{e^2a^2}{8\omega np}. \tag{4.9}$$

Similar definitions hold for ρ' and α' with the final momentum replacing the initial one. The phase-dependent momenta are given by

$$\boldsymbol{p}(\theta) = \boldsymbol{p} - e\boldsymbol{a}\cos\theta + \hat{z}I_p(\theta) \tag{4.10a}$$

$$\boldsymbol{Q}(\theta) = \boldsymbol{p}'(\theta) - \hat{z}(I_{p'}(\theta) - I_p(\theta) + p_0' - p_0) \tag{4.10b}$$

with

$$\boldsymbol{p}'(\theta) = \boldsymbol{p}' - e\boldsymbol{a}\cos\theta + \hat{z}I_{p'}(\theta). \tag{4.10c}$$

In a similar way we may expand the correction term shown in (2.23) as

$$T^{(2)} = 2\pi\sum_{l=-\infty}^{\infty}\delta\left(p_0' - p_0 - \frac{e^2a^2}{4np'} + \frac{e^2a^2}{4np} - l\omega\right)T_l^{(2)} \tag{4.11}$$

where

$$T_l^{(2)} = \int_0^{2\pi}\frac{d\theta}{2\pi}\exp[il\theta + i(\rho'-\rho)\sin\theta - i(\alpha'-\alpha)\sin 2\theta]w(\theta). \tag{4.12}$$

We find that, for the spin-$\frac{1}{2}$ case, the appropriate modification of (3.15) is

$$w(\theta) = \int d^3x\exp(i\boldsymbol{k}\cdot\boldsymbol{x})u^{(-)*}_{\boldsymbol{p}'(\theta);sc}[(-i\boldsymbol{\nabla}_{\boldsymbol{p}(\theta)} - \boldsymbol{x})\cdot e\boldsymbol{F}(\theta) + \boldsymbol{\alpha}\cdot e\hat{z}F(\theta)\cdot\boldsymbol{x}]u^{(+)}_{\boldsymbol{p}(\theta)}. \tag{4.13}$$

(A similar expression, based on (2.23*b*), is obtained for the scattering of a spin-zero particle.) Energy conservation has been used to write the exponential in the form shown in (4.13), with

$$\boldsymbol{k} \equiv \hat{z}\omega[l + (\rho'-\rho)\cos\theta - 2(\alpha'-\alpha)\cos 2\theta]. \tag{4.14}$$

We also have

$$\boldsymbol{F}(\theta) = \omega\boldsymbol{a}\sin\theta - \hat{z}\frac{\omega\boldsymbol{a}\sin\theta\cdot\boldsymbol{p}(\theta)}{np}. \tag{4.15}$$

The cross section for the absorption of l photons (or emission if l is negative) is obtained, in this variational approximation, in the form

$$\frac{d\sigma_l}{d\Omega} = (2\pi)^4 \frac{p_0}{|\boldsymbol{p}|} \sqrt{(p_0 + l\omega)^2 - m^2}\,(p_0 + l\omega) |T_l^{(1)} + T_l^{(2)}|^2. \tag{4.16}$$

4.3. Approximations of the Kroll–Watson type

Approximations to the leading term shown in (4.8) can easily be obtained, in either the intermediate- or strong-coupling regime, through an application of the methods described in subsection 4.2 to the case of a monochromatic field. Thus for the strong-coupling case the condition of stationary phase becomes

$$l + (\rho' - \rho)\cos\theta - 2(\alpha' - \alpha)\cos 2\theta = 0. \tag{4.17}$$

When this equation is combined with the energy-conservation condition, obtained from (4.7) in the form

$$l\omega = p_0' - p_0 - \frac{e^2a^2}{4np'} + \frac{e^2a^2}{4np} \tag{4.18}$$

we arrive at the relation

$$p_0' - p_0 = I_p(\theta) - I_{p'}(\theta) \tag{4.19}$$

in accordance with (4.1). Then $\boldsymbol{Q}(\theta)$, defined in (4.10*b*), may be replaced by $\boldsymbol{p}'(\theta)$ which, by virtue of (4.19), puts the t matrix on the mass shell. We shall suppose, for definiteness, that (4.17), a quadratic equation for $\cos\theta$, has one and only one physical solution $\cos\theta_0$, satisfying $|\cos\theta_0| < 1$. With the t matrix evaluated at $\theta = \theta_0$, equation (4.8) becomes

$$T_l^{(1)} \simeq J_{-l}(\rho' - \rho, \alpha' - \alpha)\, t[\boldsymbol{p}'(\theta_0), \boldsymbol{p}(\theta_0)] \tag{4.20}$$

with the generalized Bessel function defined as

$$J_{-l}(x, y) = \int_0^{2\pi} \frac{d\theta}{2\pi} \exp(il\theta + ix\sin\theta - iy\sin 2\theta). \tag{4.21}$$

By adapting the discussion given at the end of subsection 4.1 to the case of a monochromatic field we may conclude that in the intermediate-coupling regime the t matrix appearing in (4.8) may be evaluated on shell, and expressed in terms of the variables

$$p_0(\theta) \simeq p_0 + \frac{e\boldsymbol{p}\cdot\boldsymbol{a}\cos\theta}{np}$$

$$[\boldsymbol{p}'(\theta) - \boldsymbol{p}(\theta)]^2 \simeq (\boldsymbol{p}' - \boldsymbol{p})^2 + 2(p_z' - p_z)\left(\frac{\boldsymbol{p}'}{np'} - \frac{\boldsymbol{p}}{np}\right)\cdot e\boldsymbol{a}\cos\theta.$$

Here terms of second order in the amplitude of the vector potential have been dropped. This t matrix may be thought of as a function of the variable $a\cos\theta$, with the other variables temporarily suppressed. Then, to first order, we have

$$t(a\cos\theta) \simeq t(0) + a\cos\theta\, t' \tag{4.22}$$

the prime denoting differentiation. An integration by parts allows us to make the replacement

$$\cos\theta \to \cos(\theta_0) \equiv -\frac{1}{(\rho'-\rho)} \tag{4.23}$$

on the right-hand side of (4.22). The on-shell t matrix resulting from this replacement may be expressed, as in our original notation, in terms of the initial and final momenta, evaluated at the phase angle θ_0. The approximate transition amplitude is of the form shown in (4.20), where now

$$\boldsymbol{p}(\theta_0) = \boldsymbol{p} + \frac{e\boldsymbol{a}l}{(\rho'-\rho)} - \hat{\boldsymbol{z}}\frac{e\boldsymbol{p}\cdot\boldsymbol{a}l}{(\rho'-\rho)np} \qquad \boldsymbol{p}'(\theta_0) = \boldsymbol{p}' + \frac{e\boldsymbol{a}l}{(\rho'-\rho)} - \hat{\boldsymbol{z}}\frac{e\boldsymbol{p}'\cdot\boldsymbol{a}l}{(\rho'-\rho)np'}. \tag{4.24}$$

In the non-relativistic limit, with recoil corrections of order v/c ignored, we may set $\alpha'-\alpha$ equal to zero and obtain the Kroll–Watson approximation

$$T_l^{(1)} \simeq J_{-l}(\rho'-\rho)t\left(\boldsymbol{p}' - \frac{m\omega l\hat{\boldsymbol{a}}}{(\boldsymbol{p}'-\boldsymbol{p})\cdot\hat{\boldsymbol{a}}}, \boldsymbol{p} - \frac{m\omega l\hat{\boldsymbol{a}}}{(\boldsymbol{p}'-\boldsymbol{p})\cdot\hat{\boldsymbol{a}}}\right). \tag{4.25}$$

This same form applies to both intermediate- and strong-coupling regimes, but in the latter case the momentum shifts appearing in the arguments of the t matrix should be dropped for consistency; these shifts introduce corrections of first order in the frequency and the term $T_l^{(2)}$, which has been omitted in this approximation, is of the same order.

5. Summary

A trial wavefunction has been constructed which serves as a fairly accurate representation of the relativistic dynamics of a charged particle scattered by a centre of force in the presence of a strong, slowly varying external radiation field. A non-relativistic wavefunction introduced some time ago (Kroll and Watson 1973) served as a model in this construction. The effectiveness of this trial function has been enhanced, following a suggestion first made by Kaminski (1988) in a slightly different context, through its use in a variational principle for the transition amplitude. Thus, if one treats both the amplitude of the vector potential and its frequency as first-order quantities ('intermediate coupling') the error in the trial function, being proportional to the electric field, is of second order and the calculated transition amplitude contains an error of *fourth* order. The leading term in the variational expression thus obtained provides a relativistic generalization of the low-frequency approximation derived non-variationally by Kroll and Watson. The higher-order term is new. It corrects for the neglect, in the leading term, of the interaction of the projectile with the external field during the collision. This correction term contains a factor, shown in (3.15) for the spin-$\frac{1}{2}$ case, which is similar in form to the matrix element for a single-photon bremsstrahlung process. More precisely, it may be identified as the correction to the low-frequency approximation derived by Low (1958) for this process. (This identification may be established quite readily by taking the weak-field limit of the variational approximation and examining the term of first order in the electric charge.) Applications of this variational method to the development of improved low-frequency approximations for electron–atom scattering in a laser field will be reported on in the near future.

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